Isomorphic Holomorphs

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Holomorphs

Let G be a finite group.

As an abstract group, the holomorph is the semi-direct product

$$\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G)$$

where $\mathrm{Aut}(G)$ acts in the natural way on the first coordinate.

Our primary interest is the realization of the holomorph in the setting of regular representations of G in a group of permutations $B = \mathrm{Perm}(X)$.

As detailed in [4, Theorem 6.3.2], for G embedded as $\lambda(G)$ in $B = \mathrm{Perm}(G)$ we can identify

$$\mathrm{Hol}(G) = \mathrm{Norm_B}(\lambda(G)) = \rho(G)A_z \qquad (1)$$

where $\rho(G)$ is the image of the right regular representation and A_z is the stabilizer of any point $z \in G$, where $A_z \cong \mathrm{Aut}(G)$.

Recall that a permutation group $N \leq \mathrm{Perm}(X)$ is regular if it acts transitively and freely.

We observe that

$$\mathrm{Hol}(G) = \mathrm{Norm}_{\mathrm{B}}(\lambda(G)) = \mathrm{Norm}_{\mathrm{B}}(\rho(G)) \tag{2}$$

where, if G is non-abelian, we have that $\lambda(G) \neq \rho(G)$ as subgroups of B.

Beyond $\lambda(G)$ and $\rho(G)$, there are other regular subgroups of $\mathrm{Perm}(G)$ that have normalizer equal to $\mathrm{Hol}(G)$.

The ubiquitousness of regular permutation groups is further highlighted by the following important fact, which can be found in [2, Lemma 1], but also appears earlier in the literature.

Proposition

If N, N' are regular subgroups of S_n that are isomorphic as abstract groups, then they are conjugate as subgroups of S_n.

and so

$$\{\text{conjugacy classes of regular subgroups}\} \leftrightarrow \{\text{isomorphism classes}\}$$

And since

$$\mathrm{Norm_B}(\lambda(G)) = \rho(G)A_z$$

where $\rho(G) = Cent_B(\lambda(g))$ then for any regular $N \leq B = \mathrm{Perm}(X)$ we have

$$\mathrm{Norm_B}(N) = N^{opp}A_{\tau(z)} \cong \mathrm{Hol}(G)$$

where

$$N^{opp} \stackrel{def}{=} \mathrm{Cent_B}(N)$$

for $N = \tau\rho(G)\tau^{-1}$

We will use the term 'regular representation' of a group G, where $|G| = n$ to mean the image of an embedding

$$\chi(G) \leq B = \mathrm{Perm}(X)$$

for some set X where $|X| = n$, which acts regularly on X.

Clearly $\lambda(G)$ and $\rho(G)$ in $\mathrm{Perm}(G)$ are basic examples, but, again, by no means is regularity tied to left or right actions.

And indeed, where appropriate, we shall work with regular representations in S_n itself.

Even though our focus is on regular permutation groups and their normalizers, one may contemplate $\mathrm{Norm}_{\mathrm{B}}(N)$ when N is a non-regular subgroup of B.

In these situations, there is frequently a distinction between $\mathrm{Norm}_{\mathrm{B}}(N)$ as compared with $\mathrm{Hol}(N)$ although sometimes there **isn't** a distinction!

The regularity of N guarantees that $\mathrm{Norm}_{\mathrm{B}}(N) \cong \mathrm{Hol}(N)$, but when N is non-regular, the structure of the normalizer is potentially quite different than that of the abstract holomorph.

Equal vs Isomorphic Normalizers

What N and N^{opp} exemplify is that two distinct regular subgroups of B can have equal normalizers, hence obviously isomorphic holomorphs.

However, our interest is when, for two *non-isomorphic* groups G_1, G_2 of the same order, is it the case that $\mathrm{Hol}(G_1) \cong \mathrm{Hol}(G_2)$.

More broadly we consider when there exists regular representations $\mu(G_1)$ and $\nu(G_2)$ in a common symmetric group $B = \mathrm{Perm}(X)$ for which $\mathrm{Norm_B}(\mu(G_1)) = \mathrm{Norm_B}(\nu(G_2))$.

Before diving into this, we point out that regular permutation groups are of central importance in the study of Hopf-Galois structures on separable extensions, as elucidated in [3].

In particular one enumerates, for a fixed finite group Γ, regularly represented in some $B = \mathrm{Perm}(X)$, those regular $N \leq B$ of the same order as Γ, for which

$$\Gamma \leq \mathrm{Norm_B}(N)$$

where N in particular need not be isomorphic to Γ.

The totality of these, where N belongs to some particular class $[G]$ of group of the same order as Γ, is denoted $R(\Gamma, [G])$.

The relevance of isomorphic/equal holomorphs to this is that if $N \cong G_1$ is regular, where G_1 and G_2 can be regularly represented in $B = \mathrm{Perm}(X)$ so that $\mathrm{Norm_B}(\mu(G_1)) = \mathrm{Norm_B}(\nu(G_2))$ and if

$$\Gamma \leq \mathrm{Norm_B}(N)$$

one has that there exists a regular $M \leq B$ (isomorphic to G_2) for which $\mathrm{Norm_B}(M) = \mathrm{Norm_B}(N)$ and therefore

$$\Gamma \leq \mathrm{Norm_B}(M)$$

and so we have a bijection

$$R(\Gamma, [G_1]) \leftrightarrow R(\Gamma, [G_2])$$

More recently, regular permutation groups and Hopf-Galois structures are known to have an important connection to skew-braces as well, as detailed in [8], which we shall say a little bit about later on, in connection with our study of isomorphic holomorphs.

Isomorphic Holomorphs

We can remove from consideration those pairs (G_1, G_2) where one or the other are abelian, as demonstrated by Mills in [7]:

Theorem

[7, Theorem 4] If a finite abelian group G_1 and an arbitrary group G_2 have isomorphic holomorphs, then G_1 and G_2 are isomorphic.

We recall a *very* classical result, an early reference to which is given in [7], which has also been previously discussed by the presenter in [5] and [6].

Consider the dihedral and quaternionic (dicylic) groups of order $4n$ for $n \geq 3$ which we can present as

$$D_{2n} = \langle x, t \mid x^{2n} = 1, t^2 = 1, xt = tx^{-1} \rangle$$
$$Q_n = \langle x, t | \; x^{2n} = 1, t^2 = x^n, xt = tx^{-1} \rangle$$

These presentations yield identical sets of reduced words:

$$X = \{t^a x^b \ | a \in \mathbb{Z}_2, b \in \mathbb{Z}_{2n}\}$$

and they have a common automorphism group:

$$A = \{\phi_{(i,j)} \mid i \in \mathbb{Z}_{2n}, j \in U(\mathbb{Z}_{2n})\}$$
$$\text{where } \phi_{(i,j)}(t^a x^b) = t^{ia} x^{ia+jb}$$

which is naturally embedded as a subgroup of $B = \mathrm{Perm}(X)$.

Beyond this, one can show that not only do we have an isomorphism

$$\mathrm{Hol}(D_{2n}) \cong \mathrm{Hol}(Q_n)$$

as abstract groups, but also,[5, Proposition 4.10] the left regular representations of both yield the equality

$$\mathrm{Hol}(D_{2n}) = \mathrm{Hol}(Q_n)$$

as subgroups of $\mathrm{Perm}(X)$ since one can show that:

$$\rho_Q(x^b)\phi_{(i,j)} = \rho_D(x^b)\phi_{(i,j)} \tag{3}$$

$$\rho_Q(tx^b)\phi_{(i,j)} = \rho_D(tx^{b+n})\phi_{(i+n,j)} \tag{4}$$

So in particular

$$\{\rho_D(x^b), \rho_D(tx^{b+n})\phi_{(n,1)} \mid b \in \mathbb{Z}_{2n}\}$$

is a normal, **regular** subgroup of $\mathrm{Hol}(D_{2n})$ that is isomorphic to Q_n.

We make this point to show the potential distinction between

$$Hol(G_1) \cong Hol(G_2) \text{ as abstract groups}$$

vs.

$$\mathrm{Norm}_{\mathrm{B}}(\mu(G_1)) = \mathrm{Norm}_{\mathrm{B}}(\nu(G_2))$$

for regular representations μ and ν.

In particular we note this:

Proposition

Suppose that G_1 and G_2 are embedded in $B = \mathrm{Perm}(X)$ as a regular subgroups $\mu(G_1)$ and $\nu(G_2)$ where $\nu(G_2) \lhd \mathrm{Norm}_{\mathrm{B}}(\mu(G_1))$.

If $\mathrm{Hol}(G_1) \cong \mathrm{Hol}(G_2)$ as abstract groups, then $\mathrm{Norm}_{\mathrm{B}}(\nu(G_2)) = \mathrm{Norm}_{\mathrm{B}}(\mu(G_1))$.

And the converse holds too, of course.

i.e. $\mathrm{Norm}_{\mathrm{B}}(\mu(G_1)) \leq NormB(\nu(G_2))$, which implies equality

What is not immediately obvious is whether having

$$\mathrm{Hol}(G_1) \cong \mathrm{Hol}(G_2) \text{ as abstract groups}$$

implies that

$$\mathrm{Norm}_{\mathrm{B}}(\mu(G_1)) = \mathrm{Norm}_{\mathrm{B}}(\nu(G_2))$$

for regular representations of $\mu(G_1)$ and $\nu(G_2)$ in some $B = \mathrm{Perm}(X)$.

Of course, if $\mathrm{Hol}(G_1) \cong \mathrm{Hol}(G_2)$ then $\mathrm{Hol}(G_1)$ contains *at least one* normal subgroup $\widehat{G}_2$, that is isomorphic to G_2.

The issue is that, in the permutational setting,

$$\widehat{G}_2 \lhd \mathrm{Norm}_{\mathrm{B}}(G_1)$$

does not necessarily imply that $\widehat{G}_2$ is regular.

The smallest example seems to be in degree 24 which we will consider presently.

We start by observing that $\mathrm{Hol}(C_4 \times D_3) \cong \mathrm{Hol}(C_3 \rtimes D_4)$.

Consider the regular subgroup $G_1 = \langle y, x, t \rangle \cong C_4 \times D_3$ of S_{24} where

$$\begin{aligned} y &= (1,3,4,9)(2,6,7,13)(5,10,11,17)(8,14,15,20)(12,18,19,23)(16,21,22,24) \\ x &= (1,5,12)(2,16,8)(3,10,18)(4,11,19)(6,21,14)(7,22,15)(9,17,23)(13,24,20) \\ t &= (1,2)(3,6)(4,7)(5,8)(9,13)(10,14)(11,15)(12,16)(17,20)(18,21)(19,22)(23,24) \end{aligned}$$

Within $\mathrm{Norm}_{S_{24}}(G_1)$ is contained

$$\widehat{G}_2 = \langle \widehat{a}, \widehat{b}, \widehat{c} \rangle \cong C_3 \rtimes D_4$$

where

$$\begin{aligned} \widehat{a} &= (1,5,12)(2,8,16)(3,17,18,9,10,23)(4,11,19)(6,20,21,13,14,24)(7,15,22) \\ \widehat{b} &= (1,4)(2,7)(5,11)(8,15)(12,19)(16,22) \\ \widehat{c} &= (1,6)(2,3)(4,13)(5,21)(7,9)(8,18)(10,16)(11,24)(12,14)(15,23)(17,22)(19,20) \end{aligned}$$

which has the property that $\widehat{G}_2 \lhd \mathrm{Norm}_{S_{24}}(G_1)$ but, as can be seen from the cycle structure, is non-regular.

So this demonstrates that being a normal subgroup of the correct isomorphism class does *not* imply regularity.

However, $\mathrm{Norm}_{S_{24}}(G_1)$ contains another normal subgroup isomorphic to $C_3 \rtimes D_4$, which *is* actually regular.

Specifically, let $G_2 = \langle a, b, c \rangle$ where

$$a = (1,10,12,3,5,18)(2,21,8,6,16,14)(4,17,19,9,11,23)(7,24,15,13,22,20)$$
$$b = (1,9)(2,13)(3,4)(5,17)(6,7)(8,20)(10,11)(12,23)(14,15)(16,24)(18,19)(21,22)$$
$$c = (1,2)(3,13)(4,7)(5,8)(6,9)(10,20)(11,15)(12,16)(14,17)(18,24)(19,22)(21,23)$$

so that $\mathrm{Norm}_{S_{24}}(G_1) = \mathrm{Norm}_{S_{24}}(G_2)$.

So what about $\mathrm{Norm}_{S_{24}}(\widehat{G}_2)$?

Since $\widehat{G}_2 \lhd \mathrm{Norm}_{S_{24}}(G_1)$ then $\mathrm{Norm}_{S_{24}}(G_1)$ is contained in $\mathrm{Norm}_{S_{24}}(\widehat{G}_2)$.

But this is actually a proper containment, as

$$|\,\mathrm{Norm}_{S_{24}}(G_1)| = |\,\mathrm{Hol}(C_4 \times D_3)| = 576$$

while $|\,\mathrm{Norm}_{S_{24}}(\widehat{G}_2)| = 1728$.

To further highlight the subtle relationship between the abstract and permutational holomorph, consider this example:

$G_1 =$
$\langle$ (1
$(1,3,5,12)(2,7,9,18)(4,11,14,23)(6,13,16,25)(8,17,20,27)(10,19,22,29)(15,24,26,31)(21,28,30,32)$
$(1,4)(2,8)(3,11)(5,14)(6,15)(7,17)(9,20)(10,21)(12,23)(13,24)(16,26)(18,27)(19,28)(22,30)(25,31)(29,32)$
$(1,5)(2,9)(3,12)(4,14)(6,16)(7,18)(8,20)(10,22)(11,23)(13,25)(15,26)(17,27)(19,29)(21,30)(24,31)(28,32)$
$(1,6)(2,10)(3,13)(4,15)(5,16)(7,19)(8,21)(9,22)(11,24)(12,25)(14,26)(17,28)(18,29)(20,30)(23,31)(27,32)$
$\cong (C_4 \times C_2 \times C_2) \rtimes C_2$

$\widehat{G}_2 =$
$\langle(4,15)(8,21)(11,24)(14,26)(17,28)(20,30)(23,31)(27,32),$
$(1,3,5,12)(2,7,9,18)(4,11,14,23)(6,13,16,25)(8,17,20,27)(10,19,22,29)(15,24,26,31)(21,28,30,32),$
$(1,4,5,14)(2,8,9,20)(3,11,12,23)(6,15,16,26)(7,17,18,27)(10,21,22,30)(13,24,25,31)(19,28,29,32)\rangle$
$\cong C_4 \times D_4$

where now G_1 is regular with $\widehat{G_2} \lhd \mathrm{Norm}_{S_{32}}(G_1)$ where, as abstract groups, $\mathrm{Hol}(G_1) \cong \mathrm{Hol}(G_2)$.

However, not only is $\widehat{G_2}$ not regular but what is even more extraordinary is that, in fact, $\mathrm{Norm}_{S_{32}}(\widehat{G_2}) = \mathrm{Norm}_{S_{32}}(G_1)$ as subgroups of S_{32}.

We make the following definitions to highlight the (apparent) distinction between isomorphic abstract and permutational holomorphs, as well as their automorphism groups.

Definition

We say that two non-isomorphic finite groups G_1 and G_2 of the same order n have

- *equivalent regular normalizers* denoted $G_1 \sim_{\mathfrak{n}} G_2$ if there exist regular representations $\mu(G_1) \leq B$, $\nu(G_2) \leq B$ with the property that $\mathrm{Norm}_B(\mu(G_1)) = \mathrm{Norm}_B(\nu(G_2))$, where $B = \mathrm{Perm}(X)$ for $|X| = n$
- *equivalent abstract holomorphs* denoted $G_1 \sim_{\mathfrak{h}} G_2$ if $\mathrm{Hol}(G_1) \cong \mathrm{Hol}(G_2)$ as abstract groups
- *equivalent automorphism groups* denoted $G_1 \sim_{\mathfrak{a}} G_2$ if $\mathrm{Aut}(G_1) \cong \mathrm{Aut}(G_2)$ as abstract groups

Given that

$$|\operatorname{Hol}(G)| = |G| \cdot |\operatorname{Aut}(G)|$$

then if G_1 and G_2 are presumably non-isomorphic groups of the same order, where $G_1 \sim_{\mathfrak{h}} G_2$ then obviously $|\operatorname{Aut}(G_1)| = |\operatorname{Aut}(G_2)|$.

We should point out that $G_1 \sim_{\mathfrak{a}} G_2$ does not imply that $G_1 \sim_{\mathfrak{h}} G_2$.

So we ask, if $G_1 \sim_{\mathfrak{h}} G_2$ then must it be the case that $G_1 \sim_{\mathfrak{a}} G_2$?

We can say this at least in the permuational setting.

Proposition

If G_1 and G_2 are non-isomorphic groups of the same order and $G_1 \sim_{\mathfrak{n}} G_2$ then one must have that $G_1 \sim_{\mathfrak{a}} G_2$.

Proof.

For N a regular subgroup of $B = \mathrm{Perm}(X)$, we recall that $\mathrm{Norm}_{\mathrm{B}}(N) = N^{opp}A_z$ where A_z is the stabilizer in $\mathrm{Norm}_{\mathrm{B}}(N)$ of any point, which as we observed earlier is isomorphic to $\mathrm{Aut}(N)$. As such, if $\mathrm{Norm}_{\mathrm{B}}(\mu(G_1)) = \mathrm{Norm}_{\mathrm{B}}(\nu(G_2))$ then A_z must be simultaneously isomorphic to $\mathrm{Aut}(G_1)$ and $\mathrm{Aut}(G_2)$. $\square$

All of this points to an intriguing possible contrast between the abstract holomorph and the normalizer of a regular representation.

If for G_1 and G_2 of the same order we have that $G_1 \sim_{\mathfrak{h}} G_2$, but $G_1 \not\sim_{\mathfrak{a}} G_2$ then the above result implies that $G_1 \not\sim_{\mathfrak{n}} G_2$.

Circling back to Hopf-Galois theory for a moment, if $G_1 \sim_{\mathfrak{n}} G_2$ then $|R(\Gamma, [G_1])| = |R(\Gamma, [G_2])|$ but it is not clear that this is holds if merely $G_1 \sim_{\mathfrak{h}} G_2$.

The presenter is not aware, however, of any examples of pairs of groups of the same order where $G_1 \sim_{\mathfrak{h}} G_2$ but $G_1 \not\sim_{\mathfrak{n}} G_2$ or $G_1 \not\sim_{\mathfrak{a}} G_2$.

In general, our conjecture is that $G_1 \sim_{\mathfrak{h}} G_2$ implies that $G_1 \sim_{\mathfrak{n}} G_2$ although the examples above do call this into question.

Note by the way, that for the degree 32 example above, the group

$\widehat{G_2} =$
$\langle(1,10)(2,6)(3,7)(4,8)(5,22)(9,16)(11,28)(12,18)(13,19)(14,20)(15,21)(17,24)(23,32)(25,29)(26,30)(27,31),$
$(1,26,16,4)(2,20,22,21)(3,31,25,11)(5,15,6,14)(7,27,29,28)(8,10,30,9)(12,24,13,23)(17,19,32,18),$
$(1,28,16,27)(2,31,22,11)(3,20,25,21)(4,29,26,7)(5,32,6,17)(8,13,30,12)(9,24,10,23)(14,19,15,18)\rangle$

is a regular normal subgroup of $\mathrm{Norm}_{S_{32}}(G_1)$, isomorphic to G_2, and so

$$\mathrm{Norm}_{S_{32}}(G_1) = \mathrm{Norm}_{S_{32}}(\widehat{G_2})$$

Going forward, we will briefly examine the abstract equivalence $G_1 \sim_{\mathfrak{h}} G_2$ itself, but for the most part we will be focused on those examples for which we can establish that $G_1 \sim_{\mathfrak{n}} G_2$.

Starting in low order we will consider pairs (G_1, G_2) for which there exist regular representations which have equal normalizers within a given $B = \mathrm{Perm}(X)$.

Sidebar - Connection To Braces

We point out that if two non-isomorphic groups G_1 and G_2 have presentations yielding a common set of reduced words X, then we have two different group operations $G_1 \cong (X, \circ)$ and $G_2 \cong (X, \star)$.

And if $G_1 \sim_{\mathfrak{n}} G_2$ then $(X, \circ, \star)$ and $(X, \star, \circ)$ are simultaneously skew-braces, yielding, in fact, a bi-skew brace structure on X.

In particular, the fact that $G_1 \sim_{\mathfrak{n}} G_2$ implies that their common automorphism group yields an automorphism of this bi-skew brace.

Indeed, we observe (for future reference) that if $(X, \circ, \star)$ has a bi-skew brace structure, then there is a common subgroup $A \leq Aut(G_1) \cap Aut(G_2)$ for which we (likely) have an isomorphism

$$G_1 \rtimes A \cong G_2 \rtimes A$$

of the relative holomorphs of each.

The point we make is that the study of when $G_1 \sim_{\mathfrak{n}} G_2$, is a special case of the study of bi-skew braces...

or perhaps bi-skew braces yield a generalization of the study of isomorphic holomorphs, in this case the isomorphism of *relative* holomorphs.

(This is an idea still in progress...)

Examples of Isomorphic Holomorphs

The smallest example of a pair of non-isomorphic groups of equal order, with isomorphic holomorphs, are D_6 and Q_3 of order 12.

Beyond D_{2n} and Q_n there are are other pairs.

One basic method for generating examples, is to start with two groups, say D and Q of the same order m, where $\mathrm{Hol}(D) \cong Hol(Q)$.

If now we choose any group Y whose order is relatively prime to m then

$$\mathrm{Aut}(D \times Y) \cong \mathrm{Aut}(D) \times \mathrm{Aut}(Y)$$

and

$$\mathrm{Aut}(Q \times Y) \cong \mathrm{Aut}(Q) \times \mathrm{Aut}(Y)$$

which allows us to deduce that

$$\mathrm{Hol}(D \times Y) \cong \mathrm{Hol}(D) \times \mathrm{Hol}(Y) \cong \mathrm{Hol}(Q) \times \mathrm{Hol}(Y) \cong \mathrm{Hol}(Q \times Y)$$

Beyond this, there are other direct products that can be formed for which the automorphism group correspondingly splits as a direct product, even if the factors do not have relatively prime orders.

In Bidwell et. al.[1] the authors show the following.

Proposition

[1, Corollary 3.8] Let $G = H \times K$ where H and K have no common direct factor and $gcd(|H/H'|, |Z(K)|) = 1$ and $gcd(|K/K'|, |Z(H)|) = 1$ then $\mathrm{Aut}(G) = \mathrm{Aut}(H) \times \mathrm{Aut}(K)$.

Facts: $|D_{2n}/D'_{2n}| = 4$, $|Q_n/Q'_n| = 4$ and $|Z(D_{2n})| = |Z(Q_n)| = 2$ for all n.

Moreover, one has that Q_n is indecomposable for all n, D_{2n} is indecomposable for n even and that the only direct product decomposition of D_{2n} for n odd is as $D_n \times C_2$.

Thus, if Y is not D_n or C_2 then

$$gcd(4, |Z(Y)|) = 1 \text{ and } gcd(|Y/Y'|, 2) = 1$$

is sufficient to guarantee that $\mathrm{Hol}(D_{2n} \times Y) \cong \mathrm{Hol}(Q_n \times Y)$.

So for example, this obviously holds if Y is any odd order group.

The first pair which is not a D_{2n}/Q_n or a direct product thereof are to be found in degree 24.

Specifically, let G_5 and G_8 denote groups 5 and 8 in the **SmallGroups** library for groups of order 24, which we present as follows:

$$G_5 = \langle u, v, t \mid u^3, v^4, t^2, vuv^{-1}u, tut^{-1}u, tvt^{-1}v^{-1} \rangle$$
$$G_8 = \langle u, v, t \mid u^3, v^4, t^2, vuv^{-1}u, tut^{-1}u, tvt^{-1}v \rangle$$

where, in each, the subgroup $\langle u, v \rangle$ is isomorphic to $Q_3 = C_3 \rtimes C_4$, but where the difference is in the action of t on v.

The (common) automorphism group is

$$\mathcal{A} = \langle c_u, c_t, \alpha, \beta \rangle$$

where c_u and c_t denote conjugation by u and t (in both groups) and where α, β are two outer automorphisms which act as follows:

$$\begin{aligned} \alpha(u) &= u & \qquad \beta(u) &= u \\ \alpha(v) &= v^{-1} & \qquad \beta(v) &= v^{-1} \\ \alpha(t) &= v^2 t & \qquad \beta(t) &= t \end{aligned}$$

Given the presentations, we have, similar to what happens in D_{2n} and Q_n, that as sets, both G_5 and G_8 consist of a common set of reduced words $X = \{u^i v^j t^k\}$.

As such, their regular representations naturally embed as regular subgroups of $\mathrm{Perm}(X)$, and $\mathcal{A} \leq \mathrm{Perm}(X)$ as well.

And as to the equality of their holomorphs, we have that

$$\begin{aligned}\rho_{G_5}(u) &= \rho_{G_8}(u)\\ \rho_{G_5}(t) &= \rho_{G_8}(t)\\ \rho_{G_5}(v) &= \rho_{G_8}(v)\alpha\beta\end{aligned}$$

so, as subgroups of $\mathrm{Perm}(X)$, $\mathrm{Norm}_{\mathrm{B}}(G_5) = \mathrm{Norm}_{\mathrm{B}}(G_8)$, that is, $G_5 \sim_{\mathfrak{n}} G_8$.

In degree 32 there are many interesting examples, and we list them here, with their index in the **SmallGroups** library in brackets

$$(C_8 \times C_2) \rtimes C_2 \sim_{\mathfrak{n}} Q_2 \rtimes C_4 \qquad [9], [10]$$
$$D_{16} \sim_{\mathfrak{n}} Q_8 \qquad [18], [20]$$
$$C_2 \times D_8 \sim_{\mathfrak{n}} C_2 \times Q_4 \qquad [39], [41]$$
$$C_8 \rtimes (C_2 \times C_2) \sim_{\mathfrak{n}} (C_2 \times Q_2) \rtimes C_2 \qquad [43], [44]$$

and, which first happens in degree 32, a triple of groups

$$C_4 \times D_4 \sim_{\mathfrak{n}} (C_4 \times C_2 \times C_2) \rtimes C_2 \sim_{\mathfrak{n}} (C_2 \times Q_2) \rtimes C_2 \; [25], [28], [29]$$

The pair (D_{16}, Q_8) is, of course already known.

For $(C_2 \times D_8, C_2 \times Q_4)$, we again reference Bidwell et. al. who actually show the following about automorphisms of direct products.

Proposition

[1, Theorem 3.2] If H and K have no common direct factor then each $\theta \in Aut(H \times K)$ *corresponds to a matrix* $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ *where* $\alpha \in \mathrm{Aut}(H)$, $\beta \in Hom(K, Z(H))$, $\gamma \in Hom(H, Z(K))$, *and* $\delta \in \mathrm{Aut}(K)$

Thus $\mathrm{Hol}(D_8) \cong \mathrm{Hol}(Q_4)$ implies $\mathrm{Hol}(C_2 \times D_8) \cong \mathrm{Hol}(C_2 \times Q_4)$.

The groups G_5 and G_8 were be presented as split extensions $K \rtimes Q$ and $\widehat{K} \rtimes \widehat{Q}$ where $K \cong \widehat{K}$ and $Q \cong \widehat{Q}$ yielding groups on a common set X.

And their automorphism groups and holomorphs were identical as subgroups of $\mathrm{Perm}(X)$.

We can do basically the same thing with $G_9 = C_8 \rtimes (C_2 \times C_2)$ and $G_{10} = (C_2 \times Q_2) \rtimes C_2$ but instead of presenting them as split extensions with the same kernel and quotient, we can present them both as follows

$$G_9 = \langle y, x, t | y^4, x^4, t^2, [x, y]y^2, [t, y]y^2, [t, x]y \rangle$$
$$G_{10} = \langle y, x, t | y^4, x^4, t^4, [x, y]y^2, [t, y]y^2, [t, x]y, t^2y^2 \rangle$$

namely that both G_9 and G_{10} are extensions of

$$\langle y, x \mid y^4, x^4, [x, y]y^2 \rangle \cong C_4 \rtimes C_4$$

but while G_9 is a split extension of this kernel, G_{10} is not, which is similar to the difference between D_{2n} which is split, while Q_n is not.

Regardless, both G_9 and G_{10} consists of the reduced words $X = \{y^i x^j t^k\}$ and they have a common automorphism group $\mathcal{A}$ generated by the following automorphisms

	α	β	τ_1	τ_2	τ_3
y	y	y^{-1}	y	y	y
x	$y^{-1}x$	$y^{-1}x$	x	x^{-1}	x
t	yt	t	x^2t	t	y^2t

For $B = \text{Perm}(X)$ with G_9 embedded as $\rho_{G_9}(X)$ and G_{10} embedded as $\rho_{G_{10}}(X)$, the equality of their normalizers is established by verifying that

$$\rho_{G_{10}}(t) = \rho_{G_9}(t)\tau_3$$

or equivalently

$$\rho_{G_9}(t) = \rho_{G_{10}}(t)\tau_3$$

The 'triple'

$$C_4 \times D_4 \sim_{\mathfrak{n}} (C_4 \times C_2 \times C_2) \rtimes C_2 \sim_{\mathfrak{n}} (C_2 \times Q_2) \rtimes C_2 \text{ [25], [28], [29]}$$

is interesting not only because this is the first example of three groups with mutually equivalent normalizers, but it also presented a number of challenges computationally.

For a given selection of presentations, yielding a common set of reduced words, the difficulty was that the corresponding automorphism groups (while isomorphic) would not act *identically* as permutations of this set of reduced words.

For example, if we refer to these groups as G_{25}, G_{28}, and G_{29}, we have that

$$G_{25} \cong \langle x, y, z \mid x^4, y^4, z^4, x^2z^2, (xy)^2, [x, z], [y, z] \rangle$$

$$G_{28} \cong \langle x, y, z \mid x^4, y^4, z^4, x^2z^2, (xy)^2, [x, z], [y, z]z^2 \rangle$$

$$G_{29} \cong \langle x, y, z \mid x^4, y^4, z^4, x^2z^2, (xy)^2, [x, z]z^2, [y, z]z^2 \rangle$$

which results in group structures on the common set

$$X = \{x^i y^j z^k \mid i, j \in \{0 \ldots 3\}, k \in \{0, 1\}\}$$

but for these presentations, there are elements of the (mutually isomorphic) automorphism group $\mathcal{A}$ that act distinctly as permutations of X.

A successful choice we discovered was this one

$$G_{25} \cong \langle x, y, z, w \mid x^4, y^4, z^4, w^4, [x,z], [x,w], [z,w],\\ x^2w^2, [y,x]x^2, [w,y]x^2, y^2w^{-1}x, z^2(xw^{-1}), [y,z]x^2 \rangle$$

$$G_{28} \cong \langle x, y, z, w \mid x^4, y^4, z^4, w^4, [x,z], [x,w], [z,w],\\ x^2w^2, [y,x]x^2, [w,y]x^2, y^2(xw)^{-1}, z^2x^2, zyz^{-1}w^{-1}y^{-1}x^{-1} \rangle$$

$$G_{29} \cong \langle x, y, z, w \mid x^4, y^4, z^4, w^4, [x,z], [x,w], [z,w],\\ x^2w^2, [y,x]x^2, [w,y]x^2, y^2w^{-1}x, z^2x^2, zyz^{-1}w^{-1}y^{-1}x^{-1}, \rangle$$

where all are groups on the common set of words
$X = \{x^i y^j z^k w^l \mid i \in \{0 \ldots 3\}, j, k, l \in \{0, 1\}\}$.

The common automorphism group, $\mathcal{A} \cong C_2 \times (C_2^2 \wr C_2)$, is of order 128 and is group 2163 in the **AllSmallGroups** library with presentation

$$\langle a,b,c,d,e,f,g | a^2,b^2,c^2,d^2,e^2,f^2,g^2,[a,b],[a,c],[a,d],[a,e],[a,f],[a,g],$$
$$[b,c],[b,d],[b,e],[b,f],[b,g],[c,d],gcgec,[c,f],[d,e],gdgfd,$$
$$[e,f],[e,g],[f,g]\rangle$$

where the generators $\{a,\ldots,g\}$ act as follows on $\{x,y,z,w\}$:

	a	b	c	d	e	f	g
x	x	x	x^2w	w	x	x	x^3
y	xyw	x^2y	x^2y	x^2y	x^3yw	xyw	xy
z	x^3zw	x^2z	x^3zw	x^2z	z	z	z
w	w	w	x^3	x	w	w	x^2w

And as far as the normalizer equivalence, one can show the following containments of $\rho(G_{28})$ and $\rho(G_{29})$ in $\mathrm{Norm}_{\mathrm{B}}(G_{25})$:

$$\begin{aligned}
\rho_{G_{28}}(x) &= \rho_{G_{25}}(w)f \\
\rho_{G_{28}}(y) &= \rho_{G_{25}}(x^2y)bdef \\
\rho_{G_{28}}(z) &= \rho_{G_{25}}(x^3zw)aef \\
\rho_{G_{28}}(w) &= \rho_{G_{25}}(x)f
\end{aligned}$$

$$\begin{aligned}
\rho_{G_{29}}(x) &= \rho_{G_{25}}(w)f \\
\rho_{G_{29}}(y) &= \rho_{G_{25}}(y)bd \\
\rho_{G_{29}}(z) &= \rho_{G_{25}}(x^3zw)aef \\
\rho_{G_{29}}(w) &= \rho_{G_{25}}(x)f
\end{aligned}$$

and each is a normal regular subgroup of $\mathrm{Norm}_{\mathrm{B}}(G_{25})$ establishing the normalizer equivalence.

Beyond degree 32, we have calculated all other 'clusters' of groups with equivalent normalizers, up to order 60.

We present a summary of groups with isomorphic holomorphs of all orders at most 60 in Tables 1 and 2.

Two observations/conjectures:

- If n is odd then $G_1 \sim_{\mathfrak{n}} G_2$ only if $G_1 \cong G_2$.
- If $G_1 \sim_{\mathfrak{n}} G_2$ then $4 \Big| n$.

And of course our principal conjecture (question?) is that $G_1 \sim_{\mathfrak{h}} G_2$ implies $G_1 \sim_{\mathfrak{n}} G_2$.

n	$G_i \sim_{\mathfrak{n}} G_j$	SmallGroups ID's
12	$Q_3 \sim_{\mathfrak{n}} D_6$	$1, 4$
16	$Q_4 \sim_{\mathfrak{n}} D_8$	$7, 9$
20	$Q_5 \sim_{\mathfrak{n}} D_{10}$	$1, 4$
24	$Q_6 \sim_{\mathfrak{n}} D_{12}$	$4, 6$
	$C_4 \times S_3 \sim_{\mathfrak{n}} (C_6 \times C_2) \rtimes C_2$	$9, 10$
28	$Q_7 \sim_{\mathfrak{n}} D_{14}$	$1, 3$
32	$(C_8 \times C_2) \rtimes C_2 \sim_{\mathfrak{n}} Q_2 \rtimes C_4$	$9, 10$
	$Q_8 \sim_{\mathfrak{n}} D_{16}$	$20, 18$
	$C_4 \times D_4 \sim_{\mathfrak{n}} (C_4 \times C_2 \times C_2) \rtimes C_2 \sim_{\mathfrak{n}} (C_2 \times Q_2) \rtimes C_2$	$25, 28, 29$
	$C_2 \times D_8 \sim_{\mathfrak{n}} C_2 \times Q_4$	$39, 41$
	$C_8 \rtimes (C_2 \times C_2) \sim_{\mathfrak{n}} (C_2 \times Q_2) \rtimes C_2$	$43, 44$
36	$Q_9 \sim_{\mathfrak{n}} D_{18}$	$1, 4$
	$C_3 \times Q_3 \sim_{\mathfrak{n}} C_3 \times D_6$	$6, 12$
	$(C_3 \times C_3) \rtimes C_4 \sim_{\mathfrak{n}} C_2 \times ((C_3 \times C_3) \rtimes C_2)$	$7, 18$
40	$Q_{10} \sim_{\mathfrak{n}} D_{20}$	$4, 6$
	$C_4 \times D_5 \sim_{\mathfrak{n}} (C_{10} \times C_2) \rtimes C_2$	$5, 8$
44	$Q_{11} \sim_{\mathfrak{n}} D_{22}$	$1, 3$

Table : Groups with Isomorphic Holomorphs of order $n \leq 44$

n	$G_i \sim_{\mathfrak{n}} G_j$	SmallGroups ID's
48	$C_8 \times S_3 \sim_{\mathfrak{n}} C_{24} \rtimes C_2$	$4, 5$
	$Q_{12} \sim_{\mathfrak{n}} D_{24}$	$8, 7$
	$C_4 \times Q_3 \sim_{\mathfrak{n}} (C_6 \times C_2) \rtimes C_4$	$11, 19$
	$(C_3 \rtimes C_4) \rtimes C_4 \sim_{\mathfrak{n}} (C_{12} \times C_2) \rtimes C_2$	$12, 14$
	$(C_3 \times D_4) \rtimes C_2 \sim_{\mathfrak{n}} (C_3 \rtimes Q_2) \rtimes C_2 \sim_{\mathfrak{n}} (C_3 \times Q_2) \rtimes C_2 \sim_{\mathfrak{n}} C_3 \rtimes Q_4$	$15, 16, 17, 18$
	$C_3 \times D_8 \sim_{\mathfrak{n}} C_3 \times Q_4$	$25, 27$
	$CSU_2(\mathbb{F}_3) \sim_{\mathfrak{n}} GL_2(\mathbb{F}_3)$	$28, 29$
	$A_4 \rtimes C_4 \sim_{\mathfrak{n}} C_2 \times S_4$	$30, 48$
	$C_2 \times SL_2(\mathbb{F}_3) \sim_{\mathfrak{n}} ((C_4 \times C_2) \rtimes C_2) \rtimes C_3$	$32, 33$
	$C_2 \times Q_6 \sim_{\mathfrak{n}} C_2 \times D_{12}$	$34, 36$
	$C_2 \times (C_4 \times S_3) \sim_{\mathfrak{n}} C_2 \times ((C_6 \times C_2) \rtimes C_2)$	$35, 43$
	$D_4 \times S_3 \sim_{\mathfrak{n}} (C_4 \times S_3) \rtimes C_2$	$38, 39$
	$Q_2 \times S_3 \sim_{\mathfrak{n}} (C_4 \times S_3) \rtimes C_2$	$40, 41$
52	$Q_{13} \sim_{\mathfrak{n}} D_{26}$	$1, 4$
56	$Q_{14} \sim_{\mathfrak{n}} D_{28}$	$3, 5$
	$C_4 \times D_7 \sim_{\mathfrak{n}} (C_{14} \times C_2) \rtimes C_2$	$4, 7$
60	$C_5 \times (C_3 \rtimes C_4) \sim_{\mathfrak{n}} C_{10} \times S_3$	$1, 11$
	$C_3 \times (C_5 \rtimes C_4) \sim_{\mathfrak{n}} C_6 \times D_5$	$2, 10$
	$Q_{15} \sim_{\mathfrak{n}} D_{30}$	$3, 12$

Table : Groups with Isomorphic Holomorphs of order $48 \leq n \leq 60$

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